

On the Structure of the Bigrassmannian Permutation Poset

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Abstract

Let \mathfrak{S}_n and \mathfrak{B}_n denote the respective sets of ordinary and bigrassmannian permutations of order n , and let (\mathfrak{S}_n, \leq) denote the Bruhat ordering permutation poset. We extensively study the structural properties of the restricted poset (\mathfrak{B}_n, \leq) , showing among other things that it is ranked, symmetric, and possesses the Sperner property. We also give formulae for the number of bigrassmannian permutations weakly below and weakly above a fixed bigrassmannian permutation, as well as the number of maximal chains.

1 Introduction

Let $n \geq 1$ be an integer, and let $[n] := \{1, 2, \dots, n\}$. Bigrassmannian elements of a Coxeter group are elements that have exactly one left descent and exactly one right descent [7]. In this note, we focus on the symmetric group of order n permutations \mathfrak{S}_n , which is a Coxeter group of type A_{n-1} . We write a typical element $\pi \in \mathfrak{S}_n$ in one-line array notation $\pi = \pi(1)\pi(2) \cdots \pi(n)$, so that $\pi(i)$ is the image of i under π . Here the bigrassmannian (BG) permutations are those elements $\pi \in \mathfrak{S}_n$ such that π and π^{-1} admit a unique descent. Let \mathfrak{B}_n denote the set of BG permutations in \mathfrak{S}_n . Then $\pi \in \mathfrak{B}_n$ if and only if there is a triple $0 \leq a < b < c \leq n$ such that

$$\pi = 1 \cdots a(b+1) \cdots c(a+1) \cdots b(c+1) \cdots n \quad (1)$$

in one-line array notation (see, e.g., [1, Exercise 39, Page 169]). Note that $a = 0$ and $c = n$ are permitted here, and thus the initial and terminal contiguous blocks of

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respective lengths a and $n - c$ in the one-line array representation for $\pi \in \mathfrak{B}_n$ may be empty. It is now clear that $\mathfrak{b}_n := |\mathfrak{B}_n| = \binom{n+1}{3}$.

Let $\text{id}_n = 12 \cdots n \in \mathfrak{S}_n$ denote the identity permutation. In short, (1) says that every element of \mathfrak{B}_n is obtained by selecting a contiguous block of $x + y \leq n$ (with $x, y \geq 1$) entries from id_n , and interchanging the first x and last y of these entries, while preserving their respective increasing orders. In the notation of (1) we have $x = b - a$ and $y = c - b$. For example, $123784569 \in \mathfrak{B}_9$ is obtained from id_9 by interchanging the $x = 3$ entries starting at the fourth position with the next $y = 2$ entries.

Recall that the symmetric group \mathfrak{S}_n equipped with the *Bruhat order* “ \leq ” becomes a partially-ordered set (poset; see [1]). Specifically, if $\omega = \omega(1) \cdots \omega(n) \in \mathfrak{S}_n$ then a *reduction* of ω is a permutation obtained from ω by interchanging some $\omega(i)$ with some $\omega(j)$ provided $i < j$ and $\omega(i) > \omega(j)$; in other words, the location-pair (i, j) forms an *inversion* of ω . We say that $\pi \leq \sigma$ in the Bruhat order if there is a chain $\sigma = \omega_1 \rightarrow \omega_2 \rightarrow \cdots \rightarrow \omega_s = \pi$, where each ω_t is a reduction of ω_{t-1} . The number of inversions in ω_t strictly decreases with t . Indeed, one can show that if ω_2 is a reduction of ω_1 via the interchange $\omega_1(i) \leftrightarrow \omega_1(j)$, $i < j$, then

$$\begin{aligned} \text{inv}(\omega_1) &= \text{inv}(\omega_2) + 2N(\omega_1) + 1, \\ N(\omega_1) &:= |\{k : i < k < j, \omega_1(i) > \omega_1(k) > \omega_1(j)\}|; \end{aligned}$$

here $\text{inv}(\bullet)$ is the number of inversions in \bullet . The Bruhat order notion can be extended to other Coxeter groups [3], and bigrassmannian elements have been used to investigate the structure of the Bruhat order [2, 7, 9]. In this paper, we shall be chiefly interested in restricting our considerations to the bigrassmannian elements $\mathfrak{B}_n \subseteq \mathfrak{S}_n$. Figure 1 illustrates this poset of BG permutations for \mathfrak{B}_3 and \mathfrak{B}_4 .

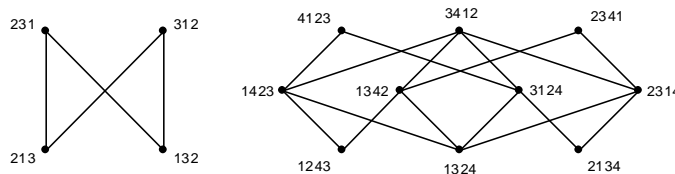


Figure 1: The posets (\mathfrak{B}_3, \leq) and (\mathfrak{B}_4, \leq) .

A large portion of this paper is devoted to studying comparable BG permutations (where comparability is inherited from (\mathfrak{S}_n, \leq)). Note well that a sequence of reductions, when starting from an element $\sigma \in \mathfrak{B}_n$, will *not necessarily* keep us within the collection of BG permutations. But checking for Bruhat comparability directly by checking all possible sequences of reductions would be an arduous task. Fortunately, there are efficient algorithms for checking Bruhat comparability that do

not rely upon this reduction operation. Each of these algorithms can be traced back to a comparability criterion due to Ehresmann [6]. The Ehresmann “tableau criterion” states that $\pi \leq \sigma$ if and only if $\pi_{i,j} \leq \sigma_{i,j}$ for all $1 \leq i \leq j \leq n$, where $\pi_{i,j}$ and $\sigma_{i,j}$ are the i th entries in the increasing rearrangement of $\pi(1), \dots, \pi(j)$ and of $\sigma(1), \dots, \sigma(j)$. These arrangements form two staircase tableaux, hence the term “tableau criterion.” For example, comparability of the BG permutations $14235 < 34512$ is verified by element-wise comparisons of the two tableaux

$$\begin{array}{cccccc} 1 & & & & & 3 \\ 1 & 4 & & & & 3 & 4 \\ 1 & 2 & 4 & & & 3 & 4 & 5 & & . \\ 1 & 2 & 3 & 4 & & 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \end{array}$$

These tableaux represent *monotone triangles* (or *Gog triangles*, in the terminology of Zeilberger [13]) formed from the two permutations. Monotone triangles are well-known to be in bijection with the collection of alternating sign matrices [4], which have been of ubiquitous combinatorial interest in recent years.

The Ehresmann tableau criterion requires that $\Theta(n^2)$ conditions be checked. However, Björner and Brenti [2] discovered an improved tableau criterion (based upon Deodhar’s more general Coxeter group characterization in [5]) that requires far fewer comparisons. For the special set of BG permutations, this improved tableau criterion requires only $O(n)$ comparisons. Indeed, given $\pi, \sigma \in \mathfrak{B}_n$, to determine whether $\pi \leq \sigma$ we need only check the row of the two tableaux that corresponds to the unique descent of π . So in our example above, $14235 < 34512$ is verified more efficiently by element-wise comparisons of the singular row

$$1 \quad 4 \qquad 3 \quad 4 \quad .$$

Recently, Kobayashi [8] studied the following question: Given a permutation $\sigma \in \mathfrak{S}_n$, how many BG permutations are weakly below σ in the Bruhat order?

Notation. Given $\sigma \in \mathfrak{S}_n$, let $\beta(\sigma)$ denote the number of $\pi \in \mathfrak{B}_n$ such that $\pi \leq \sigma$ in Bruhat order, and let $\alpha(\sigma)$ denote the number of $\pi \in \mathfrak{B}_n$ such that $\pi \geq \sigma$ in Bruhat order.

By working in the MacNeille completion of \mathfrak{S}_n (i.e., the smallest lattice that contains (\mathfrak{S}_n, \leq)), which is known to be isomorphic to the lattice of monotone triangles under entry-wise comparisons (see, e.g., Stanley [12, Exercise 7.103]), Kobayashi was able to prove the following.

Theorem 1.1 (Kobayashi [8]). *Given $\sigma \in \mathfrak{S}_n$, we have*

$$\beta(\sigma) = \frac{1}{2} \sum_{i=1}^n (\sigma(i) - i)^2.$$

By restricting to BG permutations, Kobayashi's formula simplifies considerably.

Theorem 1.2. *Given $\sigma \in \mathfrak{B}_n$, in the notation of (1) we have*

$$\beta(\sigma) = \frac{1}{2}(b-a)(c-b)(c-a).$$

An easy proof of Theorem 1.2 involves a direct computation using Theorem 1.1; we leave this to the interested reader. In Section 2 we provide an alternate proof, whose ideas are useful in finding the following parallel equation for $\alpha(\sigma)$.

Theorem 1.3. *Given $\sigma \in \mathfrak{B}_n$, in the notation of (1) we have*

$$\alpha(\sigma) = \frac{1}{2}(a+1)(n-c+1)(n-c+1+a+1).$$

We then proceed to study some of the structural properties of the poset (\mathfrak{B}_n, \leq) . All of these results hinge on the following equivalent characterization of Bruhat-comparability for BG permutations, which we prove in Section 3. A similar characterization, involving ordered triples, is given in [10]. To state the characterization, we first define a vector that encapsulates the same information contained in (1).

Definition 1.4. Let $\pi \in \mathfrak{B}_n$ with a, b , and c as in (1). We define the *lengths* of π to be $\ell_1(\pi) := a+1$, $\ell_2(\pi) := b-a$, $\ell_3(\pi) := c-b$, and $\ell_4(\pi) := n+1-c$. The *length vector* of π is $\ell(\pi) := (\ell_1(\pi), \ell_2(\pi), \ell_3(\pi), \ell_4(\pi))$. When it is unambiguous, we shall suppress the argument π and write only $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$ for the length vector.

A couple of notes are in order. First,

$$(\dagger) \quad \ell_1 + \ell_2 + \ell_3 + \ell_4 = n + 2, \text{ and}$$

$$(\ddagger) \quad \ell_i \in [n-1] \text{ for } i \in [4].$$

Furthermore any choice of ℓ_1, ℓ_2, ℓ_3 , and ℓ_4 satisfying (\dagger) and (\ddagger) corresponds to a unique element of \mathfrak{B}_n .

Theorem 1.5. *Let $\pi, \sigma \in \mathfrak{B}_n$. The following are equivalent:*

1. $\pi \leq \sigma$ and
2. $\ell_1(\pi) \geq \ell_1(\sigma), \ell_4(\pi) \geq \ell_4(\sigma), \ell_2(\pi) \leq \ell_2(\sigma), \ell_3(\pi) \leq \ell_3(\sigma)$.

The utility of Theorem 1.5 is far-reaching. Specifically, it elicits yet another alternative proof of Theorem 1.2 (which we give in Section 3). Theorem 1.3 is also rediscovered in the light of the characterization of BG permutations in Definition 1.4. We also indicate how Theorem 1.5 reveals that the Hasse diagram for (\mathfrak{B}_n, \leq) has a natural symmetry intrinsic to BG permutations in Corollary 3.3, and how this symmetry does not extend to its embedding in (\mathfrak{S}_n, \leq) .

In Section 4, we show how the main idea in the proof of Theorem 1.5 delivers a number of other structural results related to (\mathfrak{B}_n, \leq) . Among the many results there, we find the degree of a vertex in the Hasse diagram of (\mathfrak{B}_n, \leq) , as well as the total number of edges and the number of maximal (minimal) elements in this poset (in Theorems 4.1, 4.2, and 4.3, respectively). We also show that this poset is ranked and provide the corresponding level-function in Theorem 4.4, and analyze the number of chains between two comparable elements in Theorem 4.7. This latter result delivers an exact formula for the number of maximal chains in (\mathfrak{B}_n, \leq) (Theorem 4.8). Furthermore, we show that this poset possesses the Sperner property in Theorem 4.10 (i.e., there exists a rank level that is a maximum antichain; see [11]). The results of Theorems 4.11 and 4.13 describe specific properties of the poset of BG permutations that show precisely how far removed this poset is from being a lattice.

Henceforth, we shall use both (\mathfrak{B}_n, \leq) and \mathfrak{B}_n to refer to the poset of BG permutations.

2 Proofs of Theorems 1.2 and 1.3

To begin, we provide an alternate proof of Theorem 1.2. Recall the Björner-Brenti comparability criterion [2] restricted to BG permutations π and σ : to check if $\pi \leq \sigma$, we only need to check the row of the two tableaux that corresponds to the unique descent of π .

Proof of Theorem 1.2. This proof of Theorem 1.2 is independent of Theorem 1.1; in it we apply the Björner-Brenti comparison criterion. Suppose that $\sigma \in \mathfrak{B}_n$ with σ of the form given in (1). We count the $\pi \in \mathfrak{B}_n$ with $\pi \leq \sigma$ based on the position of the descent of π . Notice that by the Björner-Brenti comparison criterion, the first descent cannot happen in position i with $i \leq a$. Suppose first that the descent happens in position i , where $a+1 \leq i \leq a+c-b$ (in other words, it occurs at a place where σ has a value in the interval from $b+1$ to c). There are $(b-a)$ possible values for the i th entry of π , namely, we have $\pi(i) \in [i+1, i+b-a]$. Since $\pi \in \mathfrak{B}_n$, there can only be one position $a \leq j < i$ with $\pi(j)+1 < \pi(j+1)$. But each possible j and i meeting these conditions gives rise to a $\pi \in \mathfrak{B}_n$ with $\pi \leq \sigma$, which implies that there are $i-a$ possibilities for j . Therefore, for a fixed i with $a+1 \leq i \leq a+c-b$, we have $(i-a)(b-a)$ possible $\pi \in \mathfrak{B}_n$ with $\pi \leq \sigma$.

Now if we have $a+c-b+1 \leq i \leq c$ (in other words, where σ has a value in the interval from $a+1$ to b), then by the Björner-Brenti comparison criterion the value $\pi(i)$ has $c-i$ possible values (namely $\pi(i) \in [i+1, c]$) and for a fixed value of $\pi(i)$, there are $(c-b)$ possible $\pi \in \mathfrak{B}_n$ with $\pi \leq \sigma$ (as $\pi(j) = j$ for $1 \leq j \leq a+i-(a+c-b)$). As before, this produces $(c-i)(c-b)$ such $\pi \in \mathfrak{B}_n$. There are no $\pi \in \mathfrak{B}_n$ with $\pi \leq \sigma$ and descent at position $i > c$.

Summing these gives

$$\begin{aligned}
\sum_{i=a+1}^{a+c-b} (i-a)(b-a) + \sum_{i=a+c-b+1}^c (c-i)(c-b) &= \sum_{i=1}^{c-b} i(b-a) + \sum_{i=1}^{b-a-1} i(c-b) \\
&= (b-a) \sum_{i=1}^{c-b} i + (c-b) \sum_{i=1}^{b-a-1} i \\
&= \frac{1}{2}(b-a)(c-b)(c-a).
\end{aligned}$$

□

Example 2.1. Let $\sigma = 1562347$, so $a = 1$, $b = 4$, $c = 6$, and $n = 7$. We list the possible $\pi \in \mathfrak{B}_7$ with $\pi \leq \sigma$ based on the position i of the descent.

i	$\pi(i)$	possible π
2	5	1523467
2	4	1423567
2	3	1324567
3	6	1562347, 1263457
3	5	1452367, 1253467
3	4	1342567, 1243567
4	6	1256347, 1236457
4	5	1245367, 1235467
5	6	1235647, 1234657

Note that there are $3 = b - a$ possible π with $i = 2$, $6 = 2 \cdot 3 = (c - b)(b - a)$ with $i = 3$, $4 = 2 \cdot 2 = 2(c - b)$ with $i = 4$, and $2 = 1 \cdot 2 = (c - b)$ with $i = 5$. This gives 15 BG permutations weakly below σ . Using Theorem 1.1, we have $\frac{1}{2} \sum_{i=1}^7 (\pi(i) - i)^2 = \frac{1}{2} (0^2 + 3^2 + 3^2 + 2^2 + 2^2 + 2^2 + 0^2) = 15$.

What about $\alpha(\sigma)$? Continuing with this $\sigma = 1562347$ example, note that we find all possible $\pi \in \mathfrak{B}_7$ with $\pi \geq \sigma$ based on their first three entries, which must be larger (in increasing order) than 156. We consider both the location of the descent of π (being either at or after 3 or before 3) and the value of π prior to the descent.

$\pi(1)\pi(2)\pi(3)$	possible π
156	1567234, 1562347
456	4567123, 4561237
167	1672345
567	5671234
561	5612347
671	6712345

To illuminate α more clearly, here is a second example where we compute the number of $\pi \in \mathfrak{B}_n$ above a fixed $\sigma \in \mathfrak{B}_n$.

Example 2.2. Let $\sigma = 126734589$, so $a = 2$, $b = 5$, $c = 7$, and $n = 9$. As in the first example, we find all possible $\pi \in \mathfrak{B}_9$ based on their first three entries, separating our list based on the location of the descent of π (being either at or after 4 or before 4).

$\pi(1)\pi(2)\pi(3)\pi(4)$	possible π
1267	126789345, 126783459, 126734589
1567	156789234, 156782349, 156723489
4567	456789123, 456781239, 456712389
1278	127893456, 127834569
1678	167892345, 167823459
5678	567891234, 567812349
1289	128934567
1789	178923456
6789	678912345
1672	167234589
6712	671234589
5671	567123489
1782	178234569
7812	781234569
6781	678123459
1892	189234567
8912	891234567
7891	789123456

Notice that the table lists the 27 possible $\pi \in \mathfrak{B}_9$ with $\sigma \leq \pi$; the computation from Theorem 1.3 gives $\alpha(\sigma) = \frac{1}{2}(3)(3)(6) = 27$.

We now generalize the above examples with the proof of Theorem 1.3.

Proof of Theorem 1.3. Suppose that σ has the notation from (1). To check that $\sigma \leq \pi$ by the Björner-Brenti criterion [2], we simply need to check that the values $\sigma(1), \dots, \sigma(c - b + a)$, when written in increasing order, are smaller than the values $\pi(1), \dots, \pi(c - b + a)$ when written in increasing order. We'll count the possible π with $\sigma \leq \pi$ by considering whether the descent of π occurs prior to position $c - b + a$, or either at or after position $c - b + a$.

First, we count those π with a descent occurring at or after position $c - b + a$. Notice that once $\pi(c - b + a)$ is a fixed value with $\pi(c - b + a) \geq c$, there are $a + 1$ possible $\pi(1), \dots, \pi(c - b + a - 1)$ determined by the value of i with $\pi(i) + 1 \neq \pi(i + 1)$ (if such an i exists). Furthermore, for each fixed values $\pi(1), \dots, \pi(c + b - a)$, there are $n - \pi(c + b - a) + 1$ possible values of $\pi(c + b - a + 1), \dots, \pi(n)$, as the position of the unique descent determines the remaining values of π . Therefore there are

$(a+1)((n-c+1) + (n-c) + \cdots + 1)$ possible π with a descent at or after position $c-b+a$.

Now consider those π with a descent before position $c-b+a$. Since $\pi \in \mathfrak{B}_n$, one of the values $\pi(1), \dots, \pi(c-b+a)$ must be 1. Since the descent happens prior to position $c-b+a$, this means $\pi(c-b+a+1) < \cdots < \pi(n)$. Furthermore there is at most one i with $c-b+a+1 \leq i < n$ and $\pi(i) + 1 \neq \pi(i+1)$, which means that the values $\pi(1), \dots, \pi(c-b+a)$ must form two intervals when written in increasing order. As before, there are a possible ways to have two intervals with the lower interval containing a 1 and fixed maximum value contained in the higher interval. If the lower interval has length i ($1 \leq i \leq a$), then there are i possible ways to order the values to have a single descent — the upper interval can start at positions 1, 2, \dots , i (but not $i+1$ since the descent must occur before position $c-b+a$). Since there are $n-c+1$ possible values for π at the descent (needing to be larger than $\sigma(c-b+a)$) and $1+2+\cdots+a$ possible starting positions for the starting position of the upper interval, we have $\binom{a+1}{2}(n-c+1)$ possible $\pi \in \mathfrak{B}_n$ with $\sigma \leq \pi$ and descent of π prior to position $c-b+a$. This proves the result. \square

3 Proof of Theorem 1.5

The goal of this section is to prove Theorem 1.5 and to provide some quick consequences. To begin, recall that for $\pi \in \mathfrak{B}_n$ with a, b , and c as in (1), the length vector of π is $\ell(\pi) := (\ell_1(\pi), \ell_2(\pi), \ell_3(\pi), \ell_4(\pi))$ where $\ell_1(\pi) = a+1$, $\ell_2(\pi) = b-a$, $\ell_3(\pi) = c-b$, and $\ell_4(\pi) = n+1-c$. Moreover, we have

$$(\dagger) \quad \ell_1 + \ell_2 + \ell_3 + \ell_4 = n+2, \text{ and}$$

$$(\ddagger) \quad \ell_i \in [n-1] \text{ for } i \in [4],$$

and any choice of ℓ_1, ℓ_2, ℓ_3 , and ℓ_4 satisfying (\dagger) and (\ddagger) corresponds to a unique element of \mathfrak{B}_n . Finally note that $\beta(\pi) = \frac{1}{2}\ell_2\ell_3(\ell_2 + \ell_3)$, and $\alpha(\pi) = \frac{1}{2}\ell_1\ell_4(\ell_1 + \ell_4)$. The symmetry of the values of α and β suggests the following definition.

Definition 3.1. Let $\pi \in \mathfrak{B}_n$ have length vector $(\ell_1, \ell_2, \ell_3, \ell_4)$. Define the map $f_{2143} : \mathfrak{B}_n \rightarrow \mathfrak{B}_n$ so that $f_{2143}(\pi)$ is the element of \mathfrak{B}_n with length vector $(\ell_2, \ell_1, \ell_4, \ell_3)$.

Note that $(f_{2143})^2$ is the identity map, which implies that f_{2143} is a bijection (involution). Importantly, f_{2143} only serves as a bijection on this special set of permutations \mathfrak{B}_n ; the notion of “length vector” is meaningless outside of this context.

Remark. The similarly defined map f_{1324} corresponds to the inverse map on \mathfrak{B}_n , and f_{4321} corresponds to the conjugate map $\bar{\pi}$ (which reverses both the permutation and the rank, in other words, is defined by $\bar{\pi}(i) = n+1 - \pi(n+1-i)$). In fact, we can define the bijection f_ϕ on \mathfrak{B}_n for any $\phi \in \mathfrak{S}_4$.

Example 3.2. Consider $\pi = 126734589 \in \mathfrak{B}_9$ from Example 2.2, which has length vector $(3, 3, 2, 3)$. Then $f_{2143}(\pi)$ has length vector $(3, 3, 3, 2)$, and so $f_{2143}(\pi) = 126783459$.

Here is now the key observation, which we restate from Section 1, that reframes comparability entirely in terms of the coordinates of the length vector.

Theorem 1.5. *Let $\pi, \sigma \in \mathfrak{B}_n$. The following are equivalent:*

1. $\pi \leq \sigma$ and
2. $\ell_1(\pi) \geq \ell_1(\sigma), \ell_4(\pi) \geq \ell_4(\sigma), \ell_2(\pi) \leq \ell_2(\sigma), \ell_3(\pi) \leq \ell_3(\sigma)$.

Proof. Throughout the proof, for brevity we write $\ell_i := \ell_i(\pi)$ and $m_i := \ell_i(\sigma)$, $i \in [4]$. Notice that $\pi(i) = i$ for $1 \leq i \leq \ell_1 - 1$, $\pi(i) = i + \ell_2$ for $\ell_1 \leq i \leq \ell_1 + \ell_3 - 1$, $\pi(i) = i - \ell_3$ for $\ell_1 + \ell_3 \leq i \leq \ell_1 + \ell_2 + \ell_3 - 1$, and $\pi(i) = i$ for $\ell_1 + \ell_2 + \ell_3 \leq i \leq n$. Similarly, we have $\sigma(i) = i$ for $1 \leq i \leq m_1 - 1$, $\sigma(i) = i + m_2$ for $m_1 \leq i \leq m_1 + m_3 - 1$, $\sigma(i) = i - m_3$ for $m_1 + m_3 \leq i \leq m_1 + m_2 + m_3 - 1$, and $\sigma(i) = i$ for $m_1 + m_2 + m_3 \leq i \leq n$.

Suppose first that $\pi \leq \sigma$. Then by the Ehresmann tableau criterion the first i elements of π (in increasing order) are at most the first i elements of σ for each i . Suppose that $\ell_1 < m_1$. Then $\pi(\ell_1) > \ell_1 = \sigma(\ell_1)$, which combined with $\sigma(i) = \pi(i) = i$ for $1 \leq i < \ell_1$ creates a contradiction. Therefore $\ell_1 \geq m_1$. Next, suppose that $\ell_2 > m_2$. Then $\pi(\ell_1) = \ell_1 + \ell_2$, and $\sigma(i) \leq i + m_2 < i + \ell_2$ for all $1 \leq i \leq \ell_1$. Therefore by focusing on the first ℓ_1 terms of π and σ in increasing order we see that $\pi \not\leq \sigma$. This contradiction implies that $\ell_2 \leq m_2$. Now, recall that $\bar{\pi}$ reverses the permutation and the rank of π . Then it is easy to see, using the Ehresmann tableau criterion, that $\pi \leq \sigma$ if and only if $\bar{\pi} \leq \bar{\sigma}$. Briefly, this is because reversing the rank of the permutations reverses the original directions of the entry-wise inequalities, and then reversing the rank-reversed permutations simply puts these inequalities back to their original directions. Combining this observation with the first two inequalities gives the remaining two inequalities.

Suppose next that $\ell_1 \geq m_1$, $\ell_4 \geq m_4$, $\ell_2 \leq m_2$, and $\ell_3 \leq m_3$. We want to show that $\pi \leq \sigma$. By the Björner-Brenti criterion, we only need to show that at position $\ell_1 + \ell_3 - 1$ (the position of the unique descent of π) the entries of π in increasing order are at most the entries of σ in increasing order. The entries of π are $1, 2, \dots, \ell_1 - 1, \ell_1 + \ell_2, \ell_1 + \ell_2 + 1, \dots, \ell_1 + \ell_2 + \ell_3 - 1$. What are the first $\ell_1 + \ell_3 - 1$ entries, in order, of σ ? Since $\ell_4 \geq m_4$, we know that $\ell_1 + \ell_2 + \ell_3 \leq m_1 + m_2 + m_3$. This means that the entries for σ form two intervals, the lower starting at 1 and the higher starting at $m_1 + m_2$. If $\ell_1 + \ell_3 < m_1 + m_3$, meaning the position of the descent for σ is larger than the position of the descent for π , then the result follows from $\ell_1 \geq m_1$ and $\ell_2 \leq m_2$ (as the entries from σ are $1, \dots, m_1 - 1, m_1 + m_2, \dots, \ell_1 + \ell_3 - 1 + m_2$). If $\ell_1 + \ell_3 \geq m_1 + m_3$, then since $\ell_1 + \ell_2 + \ell_3 \leq m_1 + m_2 + m_3$ the higher interval for σ goes from $m_1 + m_2$ to $m_1 + m_2 + m_3 - 1$ and the lower interval goes from 1 to $\ell_1 + \ell_3 - m_3 - 1$. Since $m_3 \geq \ell_3$, again the result follows. But these are all possible cases, and so the theorem is proved. \square

To summarize, comparability within \mathfrak{B}_n is determined entirely by analyzing length vectors. The following are immediate consequences of Theorem 1.5, and in particular we have new proofs of Theorem 1.2 and Theorem 1.3.

Corollary 3.3. *Let $\pi, \sigma \in \mathfrak{B}_n$. Then*

- (1) $\pi \leq \sigma$ if and only if $f_{2143}(\pi) \geq f_{2143}(\sigma)$,
- (2) $\beta(\pi) = \frac{1}{2}\ell_2\ell_3(\ell_2 + \ell_3)$,
- (3) $\alpha(\pi) = \frac{1}{2}\ell_1\ell_4(\ell_1 + \ell_4)$,
- (4) $\beta(\pi) = \alpha(f_{2143}(\pi))$.

Proof. (1) is clear from Theorem 1.5. We will prove (2) only, as (3) follows from a similar argument, and the combination of (2) and (3) imply (4).

Suppose that π has length vector $(\ell_1, \ell_2, \ell_3, \ell_4)$. To find $\beta(\pi)$, we must find all elements of \mathfrak{B}_n with length vectors (m_1, m_2, m_3, m_4) so that $\ell_1 \leq m_1$, $\ell_2 \geq m_2$, $\ell_3 \geq m_3$, and $\ell_4 \leq m_4$. We imagine removing i from ℓ_2 and j from ℓ_3 and putting these on the outer two lengths; there are $i + j + 1$ ways that this can be done. Therefore

$$\beta(\pi) = \sum_{\substack{0 \leq i \leq \ell_2 - 1 \\ 0 \leq j \leq \ell_3 - 1}} (i + j + 1) = \ell_2 \binom{\ell_3}{2} + \ell_3 \binom{\ell_2}{2} + \ell_2 \ell_3 = \frac{1}{2}\ell_2\ell_3(\ell_2 + \ell_3).$$

The proof is complete. \square

As we mentioned in Section 1, item (4) of Corollary 3.3 illuminates a beautiful symmetry for the BG poset. This, when combined with Theorem 4.4 below and the fact that π and $\bar{\pi}$ are at the same rank level $\lambda(\pi) = \lambda(\bar{\pi}) := \ell_2 + \ell_3 - 2$, reveals a systematic way to envision the Hasse diagram for (\mathfrak{B}_n, \leq) . See Figure 2, where we have highlighted the *down-set* of 41235 (i.e., the set of all $\pi \in \mathfrak{B}_5$ with $\pi \leq 41235$) as well as the *up-set* of 12453 to illustrate why it is that

$$\beta(41235) = \alpha(f_{2143}(41235)) = \alpha(12453) = 6.$$

Moreover, this particular symmetry vanishes when one embeds (\mathfrak{B}_n, \leq) into (\mathfrak{S}_n, \leq) , as the level-function there is simply the number of inversions, which is entirely different from the level-function for (\mathfrak{B}_n, \leq) . For example, the incomparable BG permutations $\pi = 31245$ and $\sigma = 12453$ have respective length vectors $\ell(\pi) = (1, 2, 1, 3)$ and $\ell(\sigma) = (3, 1, 2, 1)$, and are at the same level $\lambda(\pi) = \lambda(\sigma) = 1 + 2 - 2 = 1$ in (\mathfrak{B}_5, \leq) . However, they reside at different levels in (\mathfrak{S}_5, \leq) as $\text{inv}(\pi) = 2$ and $\text{inv}(\sigma) = 1$.

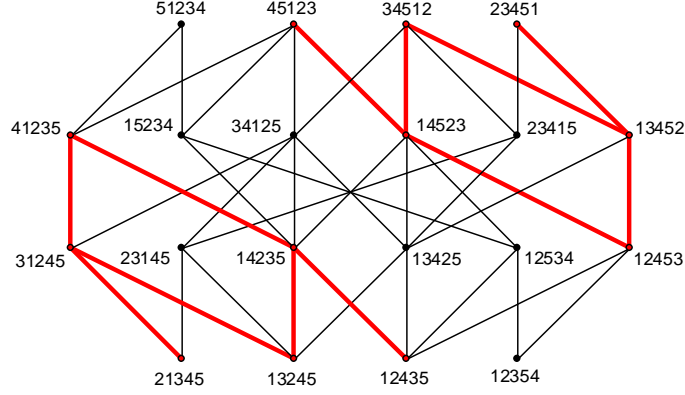


Figure 2: The poset (\mathfrak{B}_5, \leq) with “down-set” of 41235 and “up-set” of 12453 highlighted.

4 Other Structural Properties

We now explore some graph-theoretic properties concerning the Hasse diagram for (\mathfrak{B}_n, \leq) . For $\pi, \sigma \in \mathfrak{B}_n$, we say that σ *covers* π and write “ $\pi \triangleleft \sigma$ ” provided that $\pi < \sigma$ and there is no intermediate element $\rho \in \mathfrak{B}_n$ with $\pi < \rho < \sigma$. In short, σ covers π if and only if $\pi < \sigma$ and there is a Hasse arc (edge) joining these two elements.

Theorem 4.1. *Let $\delta(\pi)$ denote the degree of π in the Hasse diagram for \mathfrak{B}_n for $n > 2$. Then $\delta(\pi) \in \{2, 4, 6, 8\}$.*

Proof. Elements that π cover in the Hasse diagram have 1 subtracted from the second or third entry in the length vector and added to the first or fourth entry; elements covering π have 1 subtracted from the first or fourth entry in the length vector and added to the second or third entry. Note that all entries must be positive. \square

Theorem 4.2. *There are $4\binom{n}{3}$ edges in the Hasse diagram for \mathfrak{B}_n .*

Proof. To count the number of edges in the Hasse diagram, note that the number of “down” (i.e., covering) edges is either 4, 2, or 0; in fact it is equal to

$$4 - 2 \cdot \mathbf{1}_{\{\ell_2=1\}} - 2 \cdot \mathbf{1}_{\{\ell_3=1\}}.$$

So the total number of edges in the Hasse diagram is

$$4\binom{n+1}{3} - 2\binom{n}{2} - 2\binom{n}{2} = 4\binom{n}{3}.$$

\square

Theorem 4.3. *Both the number of minimal elements of \mathfrak{B}_n and the number of maximal elements of \mathfrak{B}_n are $n - 1$.*

Proof. An element of \mathfrak{B}_n is minimal if and only if $\ell_2 = \ell_3 = 1$. Since we have $n + 2 = \ell_1 + \ell_2 + \ell_3 + \ell_4$, the result follows. The maximal element count follows from a similar argument or by applying the function f_{2143} . \square

Theorem 4.4. *\mathfrak{B}_n is a ranked poset with level-function given by*

$$\lambda(\pi) := \ell_2 + \ell_3 - 2. \quad \square$$

Theorem 4.5. *Given $0 \leq k \leq n - 2$, there are $(k + 1)(n - k - 1)$ BG permutations at level k .*

Proof. To count the number of $\pi \in \mathfrak{B}_n$ with $k = \lambda(\pi) = \ell_2 + \ell_3 - 2$, we must count the number of ways that we could have $\ell_2 + \ell_3 = k + 2$ and $\ell_1 + \ell_4 = n - k$, with each $\ell_i \geq 1$. There are $(k + 1)$ solutions to the first equation, and $(n - k - 1)$ solutions to the second. This proves the result. \square

Remark. Theorem 4.5 provides a curious bijective proof of the classical convolution identity $\sum_{i=1}^{n-1} i(n - i) = \binom{n+1}{3}$. Indeed, the right-hand side is the number of BG permutations of order n , and the left-hand side counts these same permutations by their levels $k = i - 1$ for $1 \leq i \leq n - 1$.

Definition 4.6. A *saturated r -chain* in \mathfrak{B}_n is a set of elements $\pi_1, \dots, \pi_r \in \mathfrak{B}_n$ with $\pi_1 \triangleleft \dots \triangleleft \pi_r$.

Theorem 4.7. *Let $\pi, \sigma \in \mathfrak{B}_n$ with corresponding length vectors $(\ell_1, \ell_2, \ell_3, \ell_4)$ and (m_1, m_2, m_3, m_4) , respectively, and $\pi \leq \sigma$. Let $r = \ell_1 + \ell_4 - m_1 - m_4$. Then there are $\binom{r}{\ell_1 - m_1} \binom{r}{m_2 - \ell_2}$ saturated r -chains between π and σ .*

Proof. The saturated chains between π and σ all have length r . Furthermore, there are $\binom{r}{\ell_1 - m_1}$ ways to choose the order to remove things from ℓ_1 and ℓ_4 , and $\binom{r}{m_2 - \ell_2}$ ways to add things to m_2 and m_3 . This gives $\binom{r}{\ell_1 - m_1} \binom{r}{m_2 - \ell_2}$ saturated r -chains. \square

Theorem 4.7 enables us to count the number of *maximal chains* in \mathfrak{B}_n , i.e., the number of saturated chains connecting a minimal and maximal pair of elements in the BG poset. These are saturated $(n - 1)$ -chains in \mathfrak{B}_n , by Theorem 4.5.

Theorem 4.8. *There are 4^{n-2} maximal chains in (\mathfrak{B}_n, \leq) .*

Proof. For any pair of minimal and maximal elements $\pi, \sigma \in \mathfrak{B}_n$, with respective length vectors $\ell(\pi) = (\ell_1, 1, 1, \ell_4)$ and $\ell(\sigma) = (1, m_2, m_3, 1)$, notice that we have $\pi \leq \sigma$ by Theorem 1.5. In other words, every minimal element is dominated by

every maximal element in \mathfrak{B}_n . Thus, by Theorem 4.7 there are $\binom{n-2}{\ell_1-1} \binom{n-2}{m_2-1}$ maximal chains between π and σ , and so the total number of maximal chains equals

$$\sum_{\substack{1 \leq \ell_1 \leq n-1 \\ 1 \leq m_2 \leq n-1}} \binom{n-2}{\ell_1-1} \binom{n-2}{m_2-1} = \left(\sum_{k=0}^{n-2} \binom{n-2}{k} \right)^2 = 4^{n-2}.$$

□

We also find the distance between any two elements in the Hasse diagram.

Theorem 4.9. *Let $\pi, \sigma \in \mathfrak{B}_n$ with corresponding length vectors $(\ell_1, \ell_2, \ell_3, \ell_4)$ and (m_1, m_2, m_3, m_4) , respectively. Then the distance between π and σ in the Hasse diagram for \mathfrak{B}_n is*

$$\max\{|\ell_1 - m_1| + |\ell_4 - m_4|, |\ell_2 - m_2| + |\ell_3 - m_3|\}.$$

In particular, the Hasse diagram for \mathfrak{B}_n is connected.

Proof. Since moving along an edge in the Hasse diagram changes exactly one inner and one outer coordinate by 1, the distance is at least this maximum value. To show that this value is attained, suppose first that $|\ell_1 - m_1| + |\ell_4 - m_4|$ is the maximum value. Assume that $|\ell_1 - m_1| \geq |\ell_4 - m_4|$ and $\ell_1 > m_1$. Iteratively remove $\ell_1 - m_1$ from the first coordinate of π and add to either of ℓ_2 or ℓ_3 that is less than m_2 or m_3 , respectively (if both are at least m_2 or m_3 , respectively, then add to either coordinate). After $\ell_1 - m_1$ steps we have (m_1, n_2, n_3, ℓ_4) . Repeating this procedure with the fourth coordinate (if $\ell_4 < m_4$ we subtract from n_2 and n_3) gives the result. The case where $|\ell_2 - m_2| + |\ell_3 - m_3|$ is the maximum value is similar. □

In addition to chains, we also consider antichains in the poset \mathfrak{B}_n . This was also considered in [10] where a symmetric chain decomposition of \mathfrak{B}_n is given; we provide a proof using length vectors.

Theorem 4.10. *The poset \mathfrak{B}_n has the Sperner property, i.e., the size of the largest antichain is $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.*

Proof. We will injectively map the elements of level $i < \frac{n}{2} + 1$ (where the map uses edges in the Hasse diagram of \mathfrak{B}_n) to the elements in level $i + 1$. Consider an element with length vector $(\ell_1, \ell_2, \ell_3, \ell_4)$ in level $\ell_2 + \ell_3 - 2$.

List the elements in the level by reverse lexicographic order on ordered pairs (ℓ_2, ℓ_1) . See the first column in Figure 3 for an example when $n = 6$ in level 1.

Suppose that $\ell_2 + \ell_3 = r + 1$ and $\ell_1 + \ell_4 = s + 1$ (so $r + s = n$). For an (ℓ_2, ℓ_1) that satisfies $\ell_1 + \ell_2 > r + 1$ map $(\ell_1, \ell_2, \ell_3, \ell_4)$ to $(\ell_1 - 1, \ell_2 + 1, \ell_3, \ell_4)$. For $\ell_1 + \ell_2 \leq r + 1$ map $(\ell_1, \ell_2, \ell_3, \ell_4)$ to $(\ell_1, \ell_2, \ell_3 + 1, \ell_4 - 1)$. (Note that these conditions are equivalent

Level $\ell_2 + \ell_3 - 2 = 1$	Image in level $\ell_2 + \ell_3 - 2 = 2$
4211	3311
3212	2312
2213	<u>1313</u>
<u>1214</u>	1223
4121	3221
3122	<u>2222</u>
2123	2132
1124	1133

Figure 3: The function defined on level 1 in \mathfrak{B}_6 .

to $\ell_1 > \ell_3$ and $\ell_1 \leq \ell_3$, respectively.) See Figure 3 for an example when $n = 6$; line breaks emphasize groupings according to a fixed pair (ℓ_2, ℓ_3) .

This is a map that is defined on all $(\ell_1, \ell_2, \ell_3, \ell_4)$ with $\ell_2 + \ell_3 = r + 1$ and produces an element in the next level above. In particular, if $\ell_1 + \ell_2 > r + 1$, then $\ell_1 \neq 1$ since $\ell_3 \geq 1$ implies $\ell_2 + 1 \leq r + 1$. Similarly, if $\ell_1 + \ell_2 \leq r + 1$, then $\ell_4 \neq 1$ as $\ell_4 = 1$ means $\ell_1 = s$, and $\ell_1 + \ell_4 > \ell_2 + \ell_3$ means $s > r$ and so $\ell_1 + \ell_2 = s + \ell_2 > r + 1$. Also, the map clearly follows edges in the Hasse diagram. We need to show the map is injective. The map is clearly injective over all elements restricted to $\ell_1 + \ell_2 > r + 1$ and separately over all elements restricted to $\ell_1 + \ell_2 \leq r + 1$. Also, since the sum of the first two coordinates is invariant under the map, nothing with $\ell_1 + \ell_2 > r + 1$ can map to the same element as something with $\ell_1 + \ell_2 \leq r + 1$. Therefore the map is injective.

By symmetry, we have injective maps from a fixed level $\ell_2 + \ell_3 > \ell_1 + \ell_4$ to the level below along edges of the Hasse diagram (when n is odd, use one map between the middle levels). The edges used by the maps at each level produce disjoint paths through the Hasse diagram that meet every vertex of the Hasse diagram. Furthermore, each path intersects the middle level(s). Therefore the largest antichain is the size of the middle level, i.e., the largest antichain has size $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. \square

The poset of BG permutations is not a lattice. In fact, we can count the exact number of *lattice-obstructions* in the Hasse diagram for \mathfrak{B}_n . By this we mean a set $\{\pi_1, \pi_2, \sigma_1, \sigma_2\} \subseteq \mathfrak{B}_n$ with π_1 and π_2 in the same level k (and hence incomparable), σ_1 and σ_2 in level $k + 1$, and $\pi_i \triangleleft \sigma_j$, $i, j \in [2]$. Such a substructure is, indeed, a lattice-obstruction since σ_1 and σ_2 will have no infimum. Likewise, π_1 and π_2 will have no supremum. For brevity, let us refer to these lattice-obstructions as *butterflies*, since they resemble a butterfly; see (\mathfrak{B}_3, \leq) in Figure 1 for an example.

Theorem 4.11. *There are $\binom{n}{3} + \binom{n-2}{3}$ butterflies in the Hasse diagram for \mathfrak{B}_n . Furthermore, each Hasse edge $\pi \triangleleft \sigma$ is in either one or two butterflies. The edge $\pi \triangleleft \sigma$ is in a unique butterfly if and only if there is some fixed coordinate $i \in [4]$ such that $\ell_i(\pi) = \ell_i(\sigma) = 1$.*

Proof. Start with π_1 with length vector $(\ell_1, \ell_2, \ell_3, \ell_4)$. The σ_j must be obtained by lowering one of ℓ_1 or ℓ_4 and raising one of ℓ_2 or ℓ_3 . Since $\pi_1 \triangleleft \sigma_1$, we know that at least one of ℓ_1 and ℓ_4 is larger than 1. If exactly one of ℓ_1 or ℓ_4 is equal to 1, then without loss of generality the length vector is $(1, \ell_2, \ell_3, \ell_4)$ and note that there are exactly two elements of \mathfrak{B}_n covering π_1 . There is one possible π_2 , namely that with length vector $(2, \ell_2, \ell_3, \ell_4 - 1)$. If ℓ_1 and ℓ_4 are both larger than 1 and $\ell_2 = \ell_3 = 1$, then there are two possible π_2 . If ℓ_1 and ℓ_4 are both larger than 1 and exactly one of ℓ_2 and ℓ_3 is equal to 1, then there are three possible π_2 . If all coordinates are larger than 1, then there are four possible π_2 . See Figure 4 for the possible π_2 as well as the elements covering both π_1 and π_2 .

π_1	π_2	σ_1 and σ_2
$(1, \ell_2, \ell_3, \ell_4)$	$(2, \ell_2, \ell_3, \ell_4 - 1)$	$(1, \ell_2 + 1, \ell_3, \ell_4 - 1), (1, \ell_2, \ell_3 + 1, \ell_4 - 1)$
$(\ell_1, 1, 1, \ell_4)$	$(\ell_1 - 1, 1, 1, \ell_4 + 1)$ $(\ell_1 + 1, 1, 1, \ell_4 - 1)$	$(\ell_1 - 1, 2, 1, \ell_4), (\ell_1 - 1, 1, 2, \ell_4)$ $(\ell_1, 2, 1, \ell_4 - 1), (\ell_1, 1, 2, \ell_4 - 1)$
$(\ell_1, 1, \ell_3, \ell_4)$	$(\ell_1 - 1, 1, \ell_3, \ell_4 + 1)$ $(\ell_1 + 1, 1, \ell_3, \ell_4 - 1)$ $(\ell_1, 2, \ell_3 - 1, \ell_4)$	$(\ell_1 - 1, 2, \ell_3, \ell_4), (\ell_1 - 1, 1, \ell_3 + 1, \ell_4)$ $(\ell_1, 2, \ell_3, \ell_4 - 1), (\ell_1, 1, \ell_3 + 1, \ell_4 - 1)$ $(\ell_1 - 1, 2, \ell_3, \ell_4), (\ell_1, 2, \ell_3, \ell_4 - 1)$
$(\ell_1, \ell_2, \ell_3, \ell_4)$	$(\ell_1 - 1, \ell_2, \ell_3, \ell_4 + 1)$ $(\ell_1 + 1, \ell_2, \ell_3, \ell_4 - 1)$ $(\ell_1, \ell_2 + 1, \ell_3 - 1, \ell_4)$ $(\ell_1, \ell_2 - 1, \ell_3 + 1, \ell_4)$	$(\ell_1 - 1, \ell_2 + 1, \ell_3, \ell_4), (\ell_1 - 1, \ell_2, \ell_3 + 1, \ell_4)$ $(\ell_1, \ell_2 + 1, \ell_3, \ell_4 - 1), (\ell_1, \ell_2, \ell_3 + 1, \ell_4 - 1)$ $(\ell_1 - 1, \ell_2 + 1, \ell_3, \ell_4), (\ell_1, \ell_2 + 1, \ell_3, \ell_4 - 1)$ $(\ell_1 - 1, \ell_2, \ell_3 + 1, \ell_4), (\ell_1, \ell_2, \ell_3 + 1, \ell_4 - 1)$

Figure 4: The possible π_2 for a fixed π_1 , and the corresponding σ_1 and σ_2 . The unspecified values ℓ_i are all larger than 1.

We now need to count the number of permutations satisfying each of these cases. There are $\binom{n-1}{2}$ vectors with only $\ell_1 = 1$ and the same number with only $\ell_4 = 1$. There are $n - 3$ with only ℓ_2 and ℓ_3 being 1. There are $\binom{n-3}{2}$ with only $\ell_2 = 1$ and $\binom{n-3}{2}$ with only $\ell_3 = 1$. There are then $\binom{n-3}{3}$ with all entries larger than 1. Lastly, we divide by two for the overcount. This gives

$$\frac{1}{2} \left[2 \binom{n-1}{2} + 2 \cdot (n-3) + 3 \cdot 2 \binom{n-3}{2} + 4 \cdot \binom{n-3}{3} \right] = \binom{n}{3} + \binom{n-2}{3}$$

total butterflies in (\mathfrak{B}_n, \leq) . \square

Theorem 4.11 says that (\mathfrak{B}_n, \leq) is *far-removed* from being a lattice. In fact, *every* Hasse arc participates in a lattice-obstruction!

As a final curiosity, since \mathfrak{B}_n is not a lattice, not every collection of elements will have an infimum (supremum, respectively). This leads to the following definition.

Definition 4.12. Let $\pi_1, \dots, \pi_r \in \mathfrak{B}_n$. We call $\sigma \in \mathfrak{B}_n$ a *maximal element below* π_1, \dots, π_r if $\sigma \leq \pi_i$ for each $i \in [r]$, and if $\tau \in \mathfrak{B}_n$ satisfies $\tau \leq \pi_i$ for $i \in [r]$ and $\tau \geq \sigma$ then $\tau = \sigma$.

Note that there may be zero, one, or more than one such σ ; if there are multiple σ then they must be incomparable.

Theorem 4.13. Let $\pi_1, \dots, \pi_r \in \mathfrak{B}_n$. An element $\sigma \in \mathfrak{B}_n$ is a maximal element below π_1, \dots, π_r if and only if $\ell_1(\sigma) \geq \max_{1 \leq j \leq r} \ell_1(\pi_j)$, $\ell_2(\sigma) = \min_{1 \leq j \leq r} \ell_2(\pi_j)$, $\ell_3(\sigma) = \min_{1 \leq j \leq r} \ell_3(\pi_j)$, and $\ell_4(\sigma) \geq \max_{1 \leq j \leq r} \ell_4(\pi_j)$.

Proof. Fix $\pi_1, \dots, \pi_r \in \mathfrak{B}_n$. If σ is a maximal element below π_1, \dots, π_r , we will show that the conditions on $\ell_i(\sigma)$ for $i \in [4]$ must hold. By Theorem 1.5, it is clear that

$$\begin{aligned} \ell_1(\sigma) &\geq \max_{1 \leq j \leq r} \ell_1(\pi_j), \ell_2(\sigma) \leq \min_{1 \leq j \leq r} \ell_2(\pi_j), \\ \ell_3(\sigma) &\leq \min_{1 \leq j \leq r} \ell_3(\pi_j), \text{ and } \ell_4(\sigma) \geq \max_{1 \leq j \leq r} \ell_4(\pi_j). \end{aligned}$$

Why must there in fact be equality in the middle two coordinates? If $\ell_1(\sigma) = \ell_4(\sigma) = 1$, then for each $j \in [r]$ we have

$$\ell_1(\pi_j) = \ell_4(\pi_j) = 1 \quad \text{and} \quad \min_{1 \leq j \leq r} \ell_2(\pi_j) + \min_{1 \leq j \leq r} \ell_3(\pi_j) \leq n,$$

with equality if and only if $\pi_1 = \dots = \pi_r$. Since $\ell_2(\sigma) + \ell_3(\sigma) = n$ in this case, we have $\sigma = \pi_1 = \dots = \pi_r$, so we have equality in the middle two coordinates. On the other hand, if some outer coordinate is not 1, without loss of generality we have $\ell_1(\sigma) > 1$ and suppose that $\ell_2(\sigma) < \min_{1 \leq j \leq r} \ell_2(\pi_j)$. Then $\tau \in \mathfrak{B}_n$ with length vector $(\ell_1(\sigma) - 1, \ell_2(\sigma) + 1, \ell_3(\sigma), \ell_4(\sigma))$ has $\sigma < \tau$ and $\tau \leq \pi_j$ for $j \in [r]$.

For the converse, suppose that the conditions on $\ell_i(\sigma)$ for $i \in [4]$ hold. We know that $\sigma \leq \pi_j$ for each $j \in [r]$ by Theorem 1.5. Suppose that $\tau \in \mathfrak{B}_n$ is such that $\tau \leq \pi_j$ for $j \in [r]$ and $\sigma \leq \tau$. Then by Theorem 1.5 we have $\ell_2(\tau) \leq \ell_2(\pi_j)$ and $\ell_3(\tau) \leq \ell_3(\pi_j)$ for all $j \in [r]$, and so $\ell_2(\tau) \leq \ell_2(\sigma)$ and $\ell_3(\tau) \leq \ell_3(\sigma)$. But $\sigma \leq \tau$ implies $\ell_2(\sigma) + \ell_3(\sigma) \leq \ell_2(\tau) + \ell_3(\tau)$ by considering their respective levels. Putting these together, we have that τ and σ are in the same level; since they are comparable they must be equal. \square

In particular, if

$$\min_{1 \leq j \leq r} \ell_2(\pi_j) + \min_{1 \leq j \leq r} \ell_3(\pi_j) + \max_{1 \leq j \leq r} \ell_1(\pi_j) + \max_{1 \leq j \leq r} \ell_4(\pi_j) = n + 2 - (i - 1)$$

for some non-negative i , then there are i maximal elements below π_1, \dots, π_r . Moreover, these i elements all belong to the same level

$$\min_{1 \leq j \leq r} \ell_2(\pi_j) + \min_{1 \leq j \leq r} \ell_3(\pi_j) - 2.$$

Under the involution f_{2143} on \mathfrak{B}_n , an analogous result holds for the similarly defined *minimal element above* π_1, \dots, π_r . We leave the details of this extension to the interested reader.

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